
UNIT 13 INTRODUCTION TO OPTIMISATION

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13.0 INTRODUCTION

In previous units, we have discussed the concepts related to data and its distributions. In this unit, we discuss a very important type of problem domain called the Optimisation Problems. In the context of problem solving in computer science, optimisation means to find the best (or optimal) solution to a problem from a set of probable solutions. How will you identify the best solution?

For example, suppose you are planning to travel from Delhi to Kolkata. Your objective may be to minimise the travel time, as well as minimise the cost, given that you may use different modes of travel, such as taxi, train and air travel. Further, there may be constraints on the total cost that can be incurred by you and a maximum time in which you have to complete this journey. Thus, you need to identify different ways by which you can complete the journey with minimum cost and minimum time but under the specified constraints. This is an optimisation problem.

In this unit, you will study the mathematical representation of optimisation problems and related concepts, and a few interesting techniques that are used in data science and machine learning to address optimisation problems.

13.1 OBJECTIVES

After studying this unit, you should be able to:

- represent an optimisation problem mathematically;
- identify the maxima and minima of a mathematical representation;
- apply gradient descent and gradient ascent to a problem;
- define the stochastic gradient descent method of optimisation.

13.2 BASIC TERMINOLOGY

Before discussing the optimisation problem, we shall discuss basic definitions of some of the important terms, which are very helpful to understand the fundamentals of optimisation.

Objective Function

The objective function is the function that is to be optimised, i.e. either maximise it or minimise it. Please note that the final output of the optimisation function is the optimal value.

Variables

Different input to the objective functions are defined as the variables. Variables are represented as x_i .

Constraints

Constraints can be represented with the help of functions or equations that put limits on the variables. There are two types of constraints – equality constraints and inequality constraints.

Thus, any problem that is to be optimised can be represented using the following mathematical formulation:

Maximise or Minimise $f(x)$,
where $f(x)$ is an objective function,
 x is an optimal solution vector, such that $\{x \in S\}$
where S is the possible set of solutions.

Subject to the following constraint:

$a_i(x) \geq 0$; (inequality constraints)

$b_i(x) = 0$; (equality constraints)

$c_i(x) \geq 0$; (inequality constraints)

where $a_i(x)$, $b_i(x)$, $c_i(x)$ are the set of equality and inequality constraints on the optimisation problem

Example 1: A furniture company makes chairs and tables only. Each chair gives a profit of INR 20, whereas each table gives a profit of INR 30. Both products are processed by three machines - M1, M2 and M3. Each chair requires 3 hrs, 5 hrs, and 2 hrs on M1, M2 and M3, respectively, whereas the corresponding numbers for each table are 3 hrs, 2 hrs and 6 hrs, respectively. The machine M1 can work for 36 hrs per week, whereas M2 and M3 can work for 50 hrs and 60 hrs, respectively. How many chairs and tables should be manufactured per week to maximise the profit?

Solution: Assume x chairs and y tables are manufactured per week.

The function for the profit of the company will be:

$$P = (20x + 30y) \text{ per week.}$$

Since the objective of the company is to maximise its profit, we have to find the maximum possible value of P .

Thus, the **Objective Function** is:

$$\text{Maximise } P = 20x + 30y$$

To manufacture x chairs and y tables, the company will require $(3x + 3y)$ hrs on machine M1, but the total time available on machine M1 is 36 hrs. Therefore, we have a constraint $3x + 3y \leq 36$.

Similarly, we have the constraint $5x + 2y \leq 50$ for the machine M2 and the constraint $2x + 6y \leq 60$ for the machine M3.

Also, since it is not possible for the company to produce a negative number of chairs and tables, we have $x \geq 0$ and $y \geq 0$.

The problem, therefore, can now be written in the following format:

Objective Function: Maximise $P = 20x + 30y$

Subject to constraints: $3x + 3y \leq 36$

$$5x + 2y \leq 50$$

$$2x + 6y \leq 60$$

$$\text{and } x \geq 0, y \geq 0$$

This is a typical optimisation problem and can be solved by using linear programming. A detailed discussion on linear programming is beyond the scope of this Unit. Here, we present a graphical way to solve this problem.

We now plot the region bounded by constraints in Fig. 13.1. The shaded region is called the feasible region or the solution space, as the coordinates of any point lying in this region always satisfy the constraints. Students are encouraged to verify this by taking points (2,2), (2,4), (4,2), which lie in the feasible region. Our objective is to find the value of x and y that maximises P . You may please note that, since it is a maximisation problem, the solution lies somewhere closer to the higher points in the feasible region. These extreme points are (10, 0), (26/3, 10/3), (3, 9), and (0, 10). You can graphically check that solution may be one of the two points (26/3, 10/3) and (3, 9). You may compute the value of P for both these points as:

$$\text{For Point } (26/3, 10/3), P = 20 \cdot 26/3 + 30 \cdot 10/3 = 820/3 = 273.33$$

$$\text{For Point } (3, 9), P = 20 \cdot 3 + 30 \cdot 9 = 330$$

Thus, the company need to produce 3 chairs and 9 tables per week to maximise its profit.

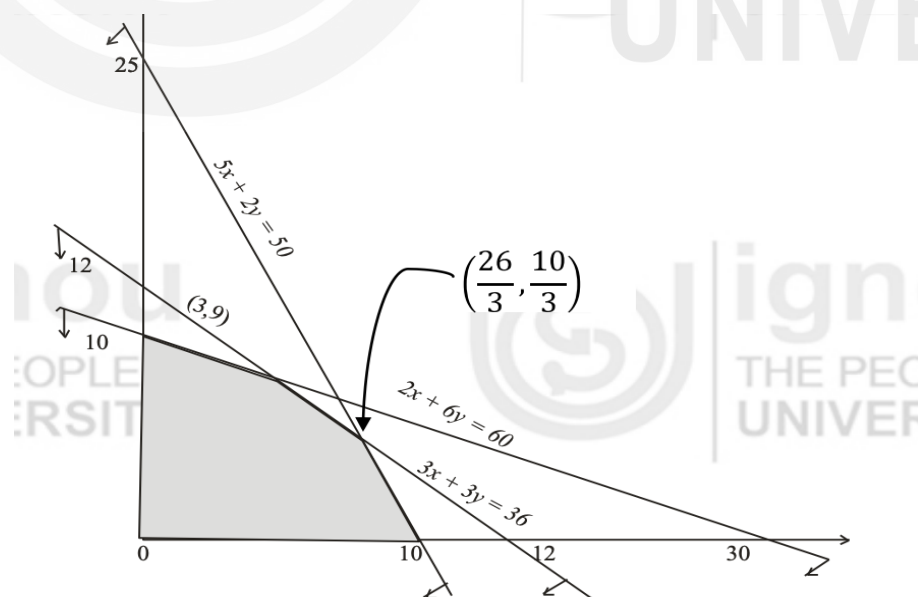


Fig. 13.1: A graphical solution to the optimisation problem

Now, let us discuss the concept of global and local optima in the context of optimisation.

13.3 GLOBAL AND LOCAL MAXIMA AND MINIMA

Several computer science problems, called optimisation problems, require you to find the maximum or the minimum value for an objective function. These extreme points, which form the solution to the problem, are called maxima or minima. For example, the Knapsack problem maximises the profit while packing a number of items in a sack/bag of finite capacity (weight), given the weight and profit of each item. On the other hand, a travelling salesman minimises the distance covered by a salesperson in their visit to several destinations.

Let us mathematically define the concept of maxima and minima. Suppose the relationship between y and x can be graphically represented by Fig. 13.2. The point at which the graph of the function stops increasing and starts declining (see point A) looks like a little hilltop, and the value that the function attains at this point is the largest it attains in its immediate vicinity. Conversely, the point on the graph (see point B) where the function stops decreasing and begins increasing looks like a little valley, and the value that the function attains at this point is the minimum in its immediate vicinity.

Definition 1: If $f(x_0) \geq f(x)$ for all x sufficiently close to x_0 , then $f(x_0)$ is said to be a relative maximum. If $f(x_0) \leq f(x)$ for all x sufficiently close to x_0 then $f(x_0)$ is said to be a relative minimum. Values that are either relative maxima or relative minima are referred to as relative extreme values or relative extrema.

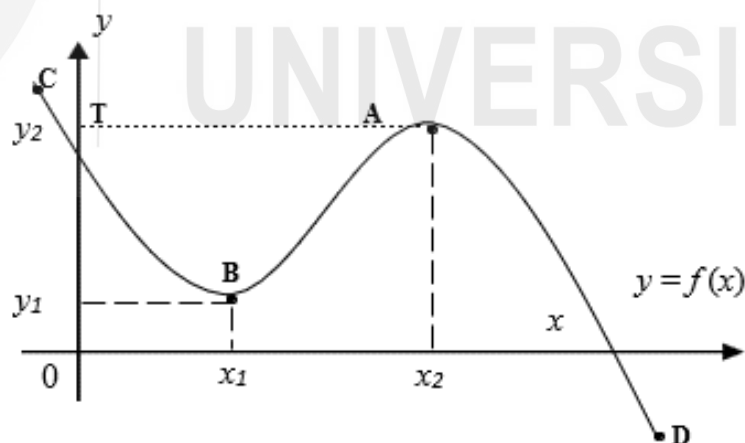


Fig. 13.2: Maxima and Minima

Attention needs to be drawn to the term 'relative'. A 'relative' or 'local' maximum need not be the 'absolute' or 'global' maximum. Similarly, a 'relative' or 'local' minimum need not be the 'absolute' or 'global' minimum. This is also evident from the Fig. 13.2. Comparison between points A (x_2, y_2) and C in Fig. 13.2, reveals that although the former falls in the category of a relative maximum, it however is not the maximum value attained by the function $y = f(x)$ overall. Similarly, compare points B (x_1, y_1) and D, where point B is a relative minimum, but not the smallest value that the function attains overall.

13.3.1 Definition

Let us first define basic terms: local minima, local maxima, global minimum and global maximum.

Local Minima (Maxima): A function $y = f(x)$ is said to have local minima (maxima) at a point $x = a$ in its domain if there exists a $\delta > 0$, such that:

$$f(x) \geq f(a) \quad (f(x) \leq f(a)) \quad \forall x \in ((a - \delta), (a + \delta))$$

For example, in Fig.13.3, the function $y = f(x)$ has local minima at the points x_2, x_4 and x_6 , while this function has local maxima at the points x_1, x_3 , and x_5 .

Global Minimum (Maximum): A function $y = f(x)$ is said to have a global minimum (maximum) at a point $x = a$ in its domain D if:

$$f(x) \geq f(a) \quad (f(x) \leq f(a)) \quad \forall x \in D$$

For example, if $D = [x_1, x_6]$, then graphically (see Fig. 13.3), we see that the global minimum of the function is at the point $x = x_2$, while the global maximum of the function is at the point $x = x_3$.

So, a global minimum means the function takes the minimum value at that point compared to the values of the function at all other points of the domain of the function. Similarly, global maximum means function takes maxima value at that point compared to the values of the function at all other points of the domain of the function. In optimisation problems we are generally interested in global minimum (maximum) instead of local minima (maxima).

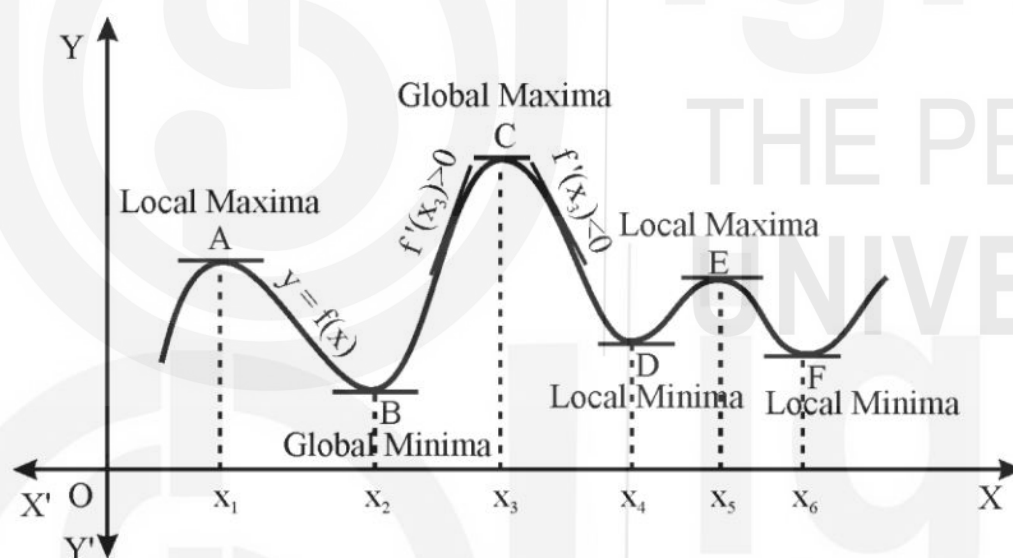


Fig. 13.3: Visualisation of Points of local and global minima and maxima

13.3.2 Slope of a Function

The relative extreme values of a function can also be characterised in terms of its slope. Assume that the total cost (C) incurred by a producer depends on his output (Q) alone. The relationship between total cost and output is represented by the inverse S-shaped curve shown in Fig. 13.4.

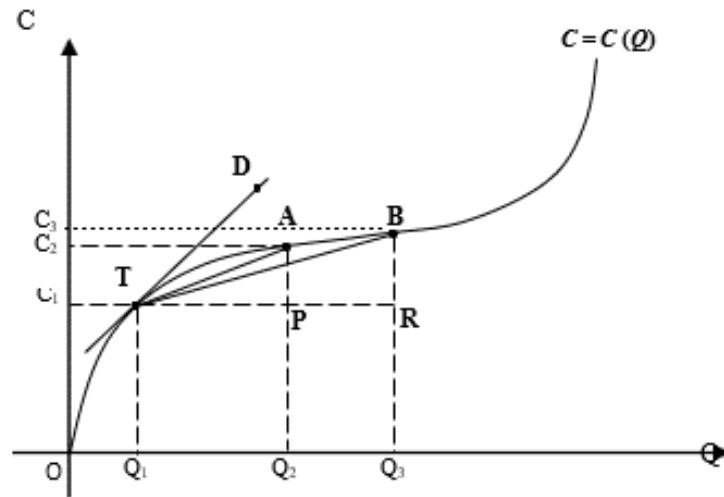


Figure 13.4: Slope of a function

Suppose, to begin with, the producer is producing OQ_1 level of output at a cost of OC_1 . This output-cost combination corresponds to point T at the total cost curve. To produce an additional output of Q_1Q_3 , the producer has to increase his cost by the amount C_1C_3 , thus $\Delta C/\Delta Q = C_1C_3 / Q_1Q_3$. Geometrically, this is the ratio of the two line segments, BR/TR and is equal to the slope of the chord TB. This ratio measures the average rate of change in cost for a particular change in output. If we vary the magnitude of change in output, reducing it to smaller margins, what happens to the total cost? For example, if the producer, instead of increasing his output to OQ_3 , increases it to only OQ_2 , then the cost increases by a lower margin of C_1C_2 . In this case $\Delta C/\Delta Q = C_1C_2 / Q_1Q_2 =$ slope of the new chord TA. Now, letting the interval, representing ΔQ grow progressively smaller (*i.e.*, $\Delta Q \rightarrow 0$), it more closely approximates a single point, in our case Q_1 . The resultant increment in output is measured from the slope of the line segment TD, *i.e.*, the tangent to the total cost curve at point T.

In other words, at Q_1 ,

$$\lim_{\Delta Q \rightarrow 0} \frac{\Delta C}{\Delta Q} = \frac{dC}{dQ} = \text{Slope of the total cost curve at point T} \\ = \text{Slope of the tangent TD}$$

In general, the slope of a function $f(x)$ at any point is equivalent to the first derivative of the function [symbolically represented as $f'(x)$] at that point and is equal to the slope of the tangent to the function at the requisite point. The first derivative of a function plays a significant role in determining the extrema of the function without even plotting it. This brings us to the second definition of relative maxima and minima, outlined in the next sub-section.

13.3.3 First Derivative Test and Relative Optima

If the first derivative of a function $f(x)$ at $x = x_0$ *i.e.*, if $f'(x_0) = 0$, then the value of the function at x_0 *i.e.*, $f(x_0)$ will be

- (i) A relative *maximum* - if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.
- (ii) A relative *minimum* - if the derivative $f'(x)$ changes its sign from negative to positive from the immediate left of the point x_0 to its immediate right.

- (iii) Neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and right of the point x_0 .

The value of the dependent variable, at which the first derivative of the function is equal to zero, *i.e.* at x_0 , is referred to as the *critical* value of x . The value of the function at its critical point *i.e.* $f(x_0)$ is known as the *stationary* value. The point with the coordinates equal to x_0 and $f(x_0)$ is accordingly called the *stationary point*.

Looking back at Fig. 13.2, it is evident that points A and B are stationary points. Point A is a relative maximum because for all values of x in the immediate left of x_2 , the function is rising (the first derivative of $f(x)$ is positive) and for all values of x in the immediate right of x_2 , the function is falling (the first derivative of $f(x)$ is negative). It is only at x_2 , the critical point, the first derivative of the function is zero and $f(x_2)$ is the corresponding stationary value. Note, the slope of the tangent to the function at A *i.e.* AT is parallel to the x -axis and is equal to zero. Analogously, one can see that point B is a point of relative minimum.

To get a clearer understanding of the process of location of extremum values of a function, let us turn to the following examples:

Example 2: Find the extremum points of the function:

$$y = 50 + 90x - 5x^2$$

Step 1: Find the first derivative of the function.

$$f'(x) = 90 - 10x \quad \text{or} \quad 10(9 - x)$$

Step 2: Equate the first derivative of the function to zero

$$10(9 - x) = 0$$

Step 3: Solve for x to obtain the critical value

The critical value of the function is:

$$x = 9$$

and the corresponding stationary value is:

$$y = f(9) = 455.$$

Let us plot the function in Example 2.

-6	-3	0	3	6	9	10	11
-670	-265	50	275	410	455	450	435

The graph of this function is shown in Figure 13.5.

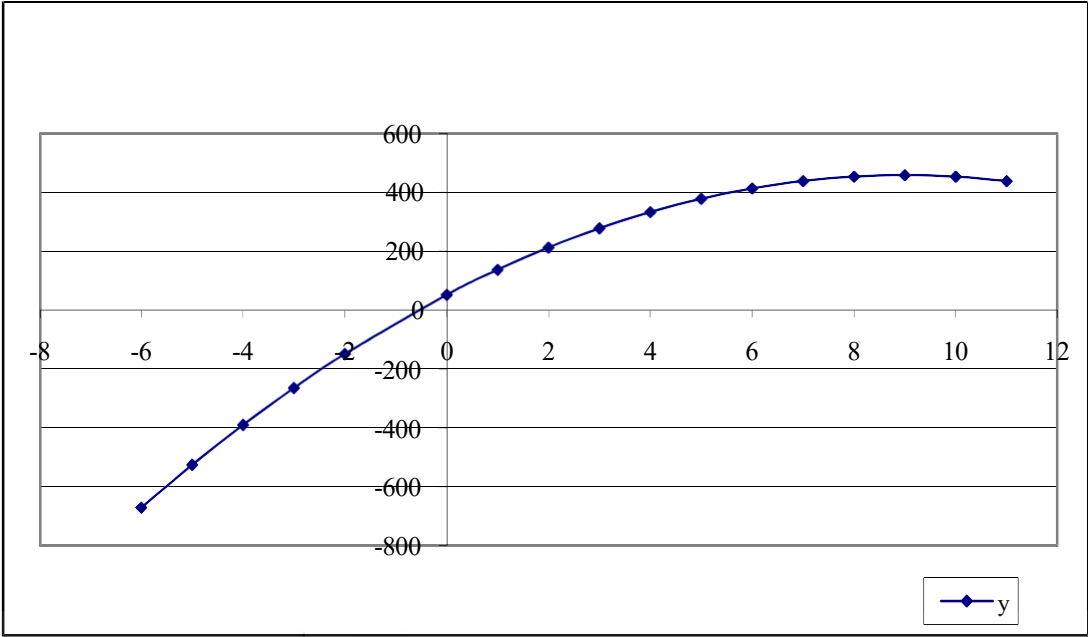


Fig. 13.5: $y = 50 + 90x - 5x^2$

It is easily verified that for all points in the neighbourhood left of $x = 9$, the function is increasing, implying that its first derivative is positive. Similarly, for all points in the immediate neighbourhood right of $x = 9$, the function is decreasing, implying its first derivative is negative. This satisfies condition (i) of the first derivative test and establishes the critical value of $x = 9$ (located in the peak of the hill!) as a relative maximum. The corresponding stationary value of the function is $y = 455$.

Example 3: Find the critical value x for the following function

$$y = x^3 - 3x + 5$$

Step1: Find the first derivative of the above function.

$$f'(x) = 3x^2 - 3$$

Step 2: Set the first derivative of the function equal to zero.

$$f'(x) = 3x^2 - 3 = 0 \quad \text{or}$$

$$3(x^2 - 1) = 0$$

Step 3: Solve for x to obtain the critical value

The critical values of x are $x = +1$ and $x = -1$, respectively.

The corresponding stationary values of y are $f(+1) = 3$ and $f(-1) = 7$, respectively.

To distinguish between the relative maximum and relative minimum, let us plot the function in Example 3.

x	-3	-2	-1	0	1	2	3
y	-13	3	7	5	3	7	23

The relationship between x and y is graphically represented in Figure 13.6.

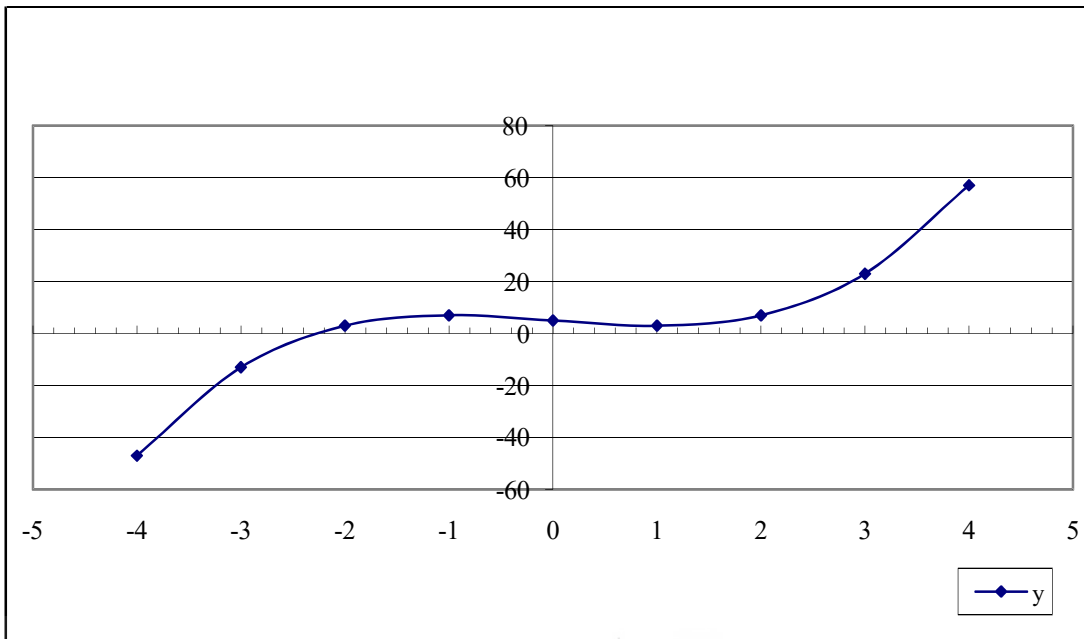


Fig. 13.6: $y = x^3 - 3x + 5$

It is easy to verify that $f'(x) > 0$ for $x < -1$ and $f'(x) < 0$ for $x > -1$ in the immediate neighbourhood of $x = -1$. Hence, the corresponding value of the function $f(-1)$ equal to 7 is established as a relative maximum. Similarly, note that $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$ in the immediate neighbourhood of $x = 1$. Consequently, the corresponding value of the function $f(1)$ equal to 3 is established as a relative minimum.

Caution: Zero slope, while necessary, is not sufficient to establish a relative extremum. If the first derivative of a function $f(x)$ is equal to zero at a value of x say x_0 , then this does not automatically ensure that x_0 is a relative extremum of the function.

To get a clearer understanding of this issue, let us turn to the example given next:

Example 4: Consider the following function whose domain is assumed to be the interval $[0, \alpha)$.

$$y = (1/3)x^3 - x^2 + x + 10$$

Differentiating with respect to x we get the first derivative as

$$x^2 - 2x + 1 \text{ or } (x-1)^2$$

Setting the first derivative of the function equal to zero yields $x = 1$ as a critical value of x . The corresponding stationary value of y is 10.33. To ascertain whether the stationary value is also a relative extremum, we have to perform the first derivative test. The graphic representation of the function in Example is as follows:

x	0	1	2	3	4	5	6
y	10	10.33	10.66	13	19.33	31.66	52

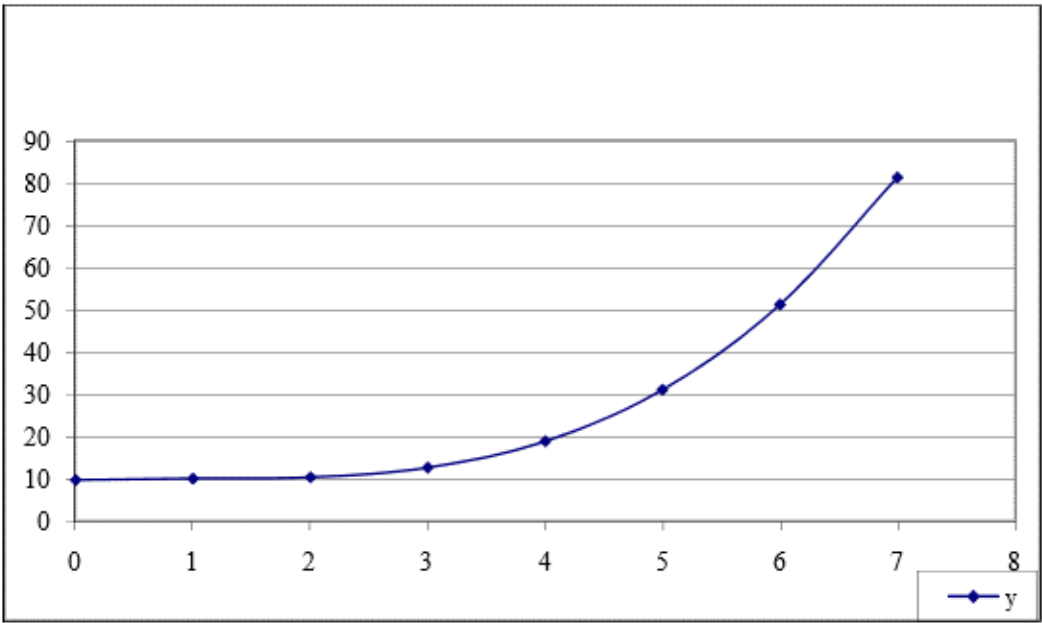


Fig. 13.7: $y = (1/3)x^3 - x^2 + x + 10$

The function attains a zero slope at the point where $x = 1$. Even though $f'(1)$ is zero—which implies $f(1)$ is a stationary value—the derivative does not change its sign from the left hand side neighbourhood of $x = 1$ to the other. In fact, as confirmed by the graph, the function is more or less flat in the immediate region of $x = 1$. On the basis of the first derivative test mentioned earlier, it can be asserted that the stationary value $f(1) = 10.33$ is neither a relative maximum nor a relative minimum.

Let us summarise, a relative extremum must be a stationary value (although the reverse is not necessarily true). To find the relative maximum or minimum of a given function, the first step would be to find the critical value of the dependent variable at which the first derivative of the function is equal to zero. This will enable us to find the stationary values of the function. To ascertain whether the stationary value is also a relative maximum or relative minimum, one needs to apply the first derivative test.

13.3.4 The Problem of Non-Differentiability

So far, we have assumed that the function is a continuous, differentiable function. In this section, we will look into two cases where this restrictive assumption is relaxed.

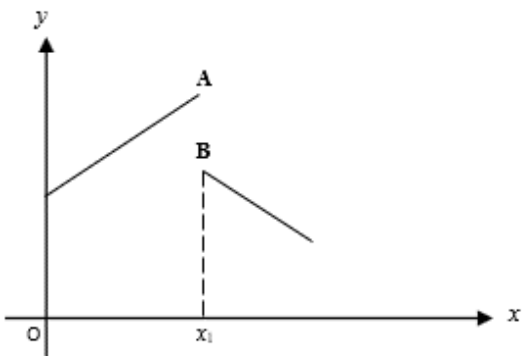


Fig. 13.8: A Discontinuous Function

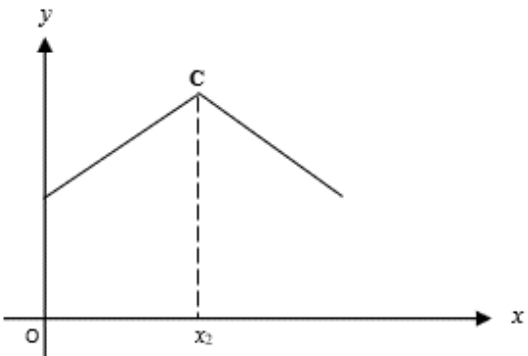


Fig. 13.9: Function with a Kink

In Fig. 13.8 and 13.9, we represent relationships between x and y that exhibit two important types of irregularities. In Fig. 13.8, the function is discontinuous at x_1 . At x_1 , the graph has a complete break or discontinuity AB. The difficulty is that in this discontinuous stretch, the first derivative of the function is not even defined. It is not possible to draw a unique tangent to the curve at these points. However, note at A, the function attains a maxima *i.e.*, at x_1 the dependent variable y attains the largest possible value. A similar kind of problem is also encountered in case of the function exhibited in Fig. 13.9. In this case, the graph of the function has a kink at point C corresponding to x_2 . The first derivative of the function is not defined at the kink and there is no unique tangent to the curve at x_2 . Once again note that the function attains a maxima at point C corresponding to x_2 . In other words, in the presence of discontinuities and kinks the derivative is not defined and hence it is not possible to employ the maximisation criteria outlined earlier.

Hitherto we have only considered first derivative of a function. In the next subsection we will define the second derivative of the function and subsequently see the role played by it in determining the relative extrema of a function.

13.3.5 The Second-Order Derivative and Condition for Optima

Assuming that the first derivative $f'(x)$ is itself a function of x , the second derivative of the function is obtained by differentiating this function again with respect to x . Symbolically, the second derivative is represented as $f''(x)$. The double prime indicates that the function $y = f(x)$ has been differentiated twice with respect to x . The expression (x) following the double prime indicates that the second derivative is also a function of x . If the second derivative $f''(x)$ exists for all values in the domain, the function $f(x)$ is said to be twice differentiable; if, in addition, $f''(x)$ is continuous, the function $f(x)$ is said to be twice continuously differentiable.

Example 5: Find the second derivative of the following function

$$y = f(x) = 4x^3 + 5x^2 - 3x + 10$$

Step 1: Differentiate the equation in Example 5 with respect to x to find the first derivative. We obtain the following equation:

$$f'(x) = 12x^2 + 10x - 3$$

Step 2: Now differentiate this equation with respect to x to obtain the second derivative of the original function:

$$f''(x) = 24x + 10$$

The first derivative of the function *i.e.*, $f'(x)$ measures the slope of the function or the rate of change of the function. If the first derivative is positive, *i.e.*, if $f'(x) > 0$, then the function is increasing; and if the derivative is negative *i.e.*, if $f'(x) < 0$, then the function is decreasing. Analogously, the second derivative *i.e.*, $f''(x)$ measures the rate of change of the first derivative $f'(x)$. If the second derivative is positive *i.e.*, if $f''(x) > 0$, then there is an increasing rate of change; and when $f''(x) < 0$, then the rate of change is decreasing. In other words, the second derivative measures *the rate of change of the rate of change of the original function* $f(x)$. Note that if $f'(x) > 0$ and $f''(x) > 0$, then this means that the function has a positive slope which is changing at an increasing rate. In other

words, the function is said to be increasing at an increasing rate. Conversely, if $f'(x) < 0$ and $f''(x) < 0$, then this means that the function has a negative slope which is changing at a decreasing rate. In other words, the function is said to be decreasing at a decreasing rate.

The Second Order Derivative Test

This test uses the second derivative of the function in question, hence the name.

Assume that $f'(x_0) = 0$, (so x_0 is a critical point!), then

- 1) If $f''(x_0) > 0$, then $f(x_0)$ is a relative minimum value.
- 2) If $f''(x_0) < 0$, then $f(x_0)$ is a relative maximum value.

As mentioned earlier, the zero slope condition, in other words $f'(x) = 0$ at $x = x_0$, was deemed to be a 'necessary condition' for $f(x_0)$ to be a relative extremum. Since this is based on the first derivative of the function, it is also known as the *first-order-condition*. Once we verify that the first order condition is satisfied, the negative (positive) sign of $f''(x)$ at $x = x_0$ is sufficient to ensure that x_0 corresponds to a relative maximum (minimum). Since the sufficiency condition is based on the second derivative of the function, it is also referred to as the *second-order-condition*.

To get a clearer understanding of how the second derivative enables us to determine whether the stationary value is a '*relative maximum*' or a '*relative minimum*', let us look back at Fig. 13.2. Recall, the extreme values of this function lies in the level stretch, either in the bottom of the hill (point B) or at the peak of the hill (point A). Around the point on the graph corresponding to the relative maximum value (point A), the graph is concave down. This makes sense, since A (x_2, y_2) is the highest point on the graph in an interval around x_2 , so the graph must 'bend down' away from the peak, making it concave down. If we pick any two points in this region of the graph, then the straight line joining these two points will lie entirely below the graph except for the two end points on the curve.

This ensures that at x equal to x_2 , $f''(x_2) < 0$ is satisfied. Similarly, around the point B (x_1, y_1) on the graph in Fig. 13.2, the graph is convex up. Once again, this makes sense. This point is the lowest point on the graph in an interval around x_1 , so the graph has to 'bend up' away from the point (x_1, y_1) , making it convex up. If we pick any two points in this region of the graph, then the straight line joining these two points will lie entirely above the graph except for the two end points on the curve. This ensures that at x equal to x_1 , $f''(x_1) > 0$ is satisfied.

Let us apply this test to the function in example earlier. We have the critical points $x = +1$ and $x = -1$, and the first derivative, $f'(x) = 3x^2 - 3$. The second derivative is then $f''(x) = 6x$. Applying the second derivative test to our two critical points we find that $f''(1) = +6 (> 0)$, making $f(1) = 3$ a relative minimum and $f''(-1) = -6 (< 0)$ making $f(-1) = 7$ a relative maximum.

Similarly, the second order condition proves to be a very useful test to determine whether the stationary value obtained in the earlier example corresponds to a relative extremum without actually plotting the function. In this case, we obtained the first derivative, $f'(x) = x^2 - 2x + 1$ and the critical point at $x = +1$. The second derivative is then $f''(x) = 2x - 2$. Applying the second derivative test to our critical point, we find that $f''(+1) = 2 - 2 = 0$. In other words, the *second-order condition* does not hold at the critical point $x = +1$, establishing

that the stationary value $f(+1) = 10.33$ is neither a relative maximum nor a relative minimum. Maxima and minima can also be used for two or more independent variables; in this case, partial differentiation is required. However, a detailed discussion on this topic is beyond the scope of this Unit.

Check Your Progress 1

- 1) Find the maximum and minimum values of

(a) $y = x^3 - 3x^2 + 2$

(b) $y = 3x^4 - 4x^3 - 12x^2 + 2$

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- 2) Find the relative maxima and minima of y by the second-derivative test:

(a) $y = \frac{1}{3}x^3 - 3x^2 + 5x + 33$

(b)

$$y = \frac{(x-1)^2}{x} \quad x \neq 0$$

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13.4 GRADIENT DESCENT AND ASCENT

Sometimes, analytic methods do not work to solve optimisation problems. So, we need some method that, if not exact, at least can provide an approximate solution. The gradient descent method is one such method that is discussed in this section. But, before we discuss the gradient descent method, let us describe some terms.

Suppose we have a function:

$$w = f(x_1, x_2, x_3, \dots, x_n)$$

then **the Gradient of this function** is given by (using partial derivatives)

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \dots(13.1)$$

Similarly, the **Jacobian of f** is a row vector given by

$$J_f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \dots (13.2)$$

$$\text{From (13.1) and (13.2) note that } J_f = (\nabla f)^T \quad \dots (13.3)$$

Hessian matrix is an n by n matrix formed by all second-order partial derivatives and is given by

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \quad \dots (13.4)$$

Taylor's theorem for this can be written as follows.

$$\begin{aligned} f(a + h_1, b + h_2) = f(a, b) &+ \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}_{(a,b)} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &+ \frac{1}{2} [h_1 \ h_2] \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}_{(a,b)} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \dots \end{aligned} \quad \dots (13.5)$$

$$\text{Or } f(a + h_1, b + h_2) = f(a, b) + \nabla^T f(X_1)h + \frac{1}{2} h^T H_f(X_1)h + \dots \quad \dots (13.6)$$

$$\text{where } \nabla f(X_1) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \text{ evaluated at the point } X_1 = (a, b), h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \text{ and} \quad \dots (13.7)$$

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}_{(a,b)} = \text{value of Hessian matrix evaluated at the point } (a, b) \quad \dots (13.8)$$

The gradient descent method is an iterative method. You know that in iterative methods, we start with some initial guess and improve the solution step by

step. To obtain an updated solution from an initial guess, we need some formula. Such a formula for the gradient descent method is given as follows.

$$X_{k+1} = X_k + \lambda_k (-\nabla f(X_k)), \quad k = 0, 1, 2, 3, \dots \quad \dots (13.9)$$

where

X_0 = initial guess

λ_k = step size at k^{th} iteration

Now, one important question that may arise and should arise in your mind is when we should stop the iterative process of this method.

The following are the rules that can be used as a stopping criterion. Suppose ε_0 be a given number, then:

Rule 1: We can choose to stop when $\|f(X_{k+1}) - f(X_k)\| < \varepsilon_0$.

Rule 2: We can choose to stop when $\|X_{k+1} - X_k\| < \varepsilon_0$.

Rule 3: Related to gradient: We can choose to stop when

- $\nabla f(X_k) = O = \text{Zero vector}$
- Or $\|\nabla f(X_k)\| < \varepsilon_0$

Another important question that may arise in your mind, and, in fact, should arise is: How does this method work? Keeping the applied nature of this programme in view, let us answer this question intuitively.

- You know that the inner product (dot product) of a vector u is maximum in the direction of the vector u itself.
- The gradient of a function at any point is also a vector.
- Directional derivatives of a function at a point along a vector u is given by taking the inner product (dot product) of the gradient vector with a unit vector along the vector u .
- Keeping the points given above in view, the maximum ascent (steepest up direction) direction on the surface of a function f at a point X_0 is along the vector $\nabla f(X_0)$, and the maximum descent (steepest down direction) direction on the surface of a function f at a point X_0 is along the vector $-\nabla f(X_0)$.

Now, the equation (13.9), i.e.,

$$X_{k+1} = X_k + \lambda_k (-\nabla f(X_k)), \quad k = 0, 1, 2, 3, \dots$$

states that to choose the value of X_{k+1} , we start from the point X_k and move along the steepest downward direction (because here we have $-\nabla f(X_0)$) till the

point where the moving direction remains steepest descent. How can we obtain such a point? This point can be obtained by optimising the function f along that particular direction, with respect to λ_k , i.e., we can choose λ_k such that

$$\frac{d}{d\lambda_k}(f(X_{k+1})) = 0 \quad \dots (13.10)$$

We continue the above iterative process till the time given stopping criterion is met. This completes the answer to the question of how this method works.

Remark: Using the Taylor approximation of the function f , we can obtain a result for obtaining the value of λ_k as

$$\begin{aligned} \Rightarrow \lambda_k &= \frac{-\nabla^T f(X_k) S_k}{S_k^T H_f(X_k) (\lambda_k S_k)} \\ \Rightarrow \lambda_k &= \frac{S_k^T S_k}{S_k^T H_f(X_k) S_k} \text{ as } S_k = -\nabla f(X_k) \end{aligned} \quad \dots (13.11)$$

So, finally, before explaining this method with the help of an example, let us list steps that you can follow for applying this method in examples.

Suppose we want to find a point of local minimum of the function f by starting from the initial point X_0 .

Step 1: Make an initial guess of the solution and denote it by X_0 .

Step 2: Obtain the value of the gradient of f at the point X_0 and set $S_0 = -\nabla f(X_0)$

Step 3: Obtain the value of Hessian matrix of f at the point X_0 and denote it by $H_f(X_0)$.

Step 4: Obtain the value of λ_0 using the expression $\lambda_0 = \frac{S_0^T S_0}{S_0^T H_f(X_0) S_0}$

Step 5: Obtain the value of X_1 in terms of X_0 , λ_0 , and $\nabla f(X_0)$ using the relation $X_1 = X_0 - \lambda_0 \nabla f(X_0)$, which can be obtained by putting $k = 0$ in (13.9).

Step 6: Check whether the stopping criterion is met. If the stopping criterion is met, then stop and declare that X_1 as the optimal solution. If the stopping criterion is not met, then repeat the above process and obtain X_2 . Continue till the time the stopping criterion is met.

Let us explain the procedure through the following example.

Example 6: Using the steepest descent method, obtain local minima of the function $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 2x_2$ by taking an initial guess as the point $(0, 0)$. The stopping criterion is $\nabla f(X_k) = \mathbf{0} = \text{Zero vector}$.

Solution: The given function is $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 2x_2 \quad \dots (13.12)$

First of all, let us visualise this function in Fig. 13.10 to get an idea of approximately where the point(s) of local minima lies(lie).

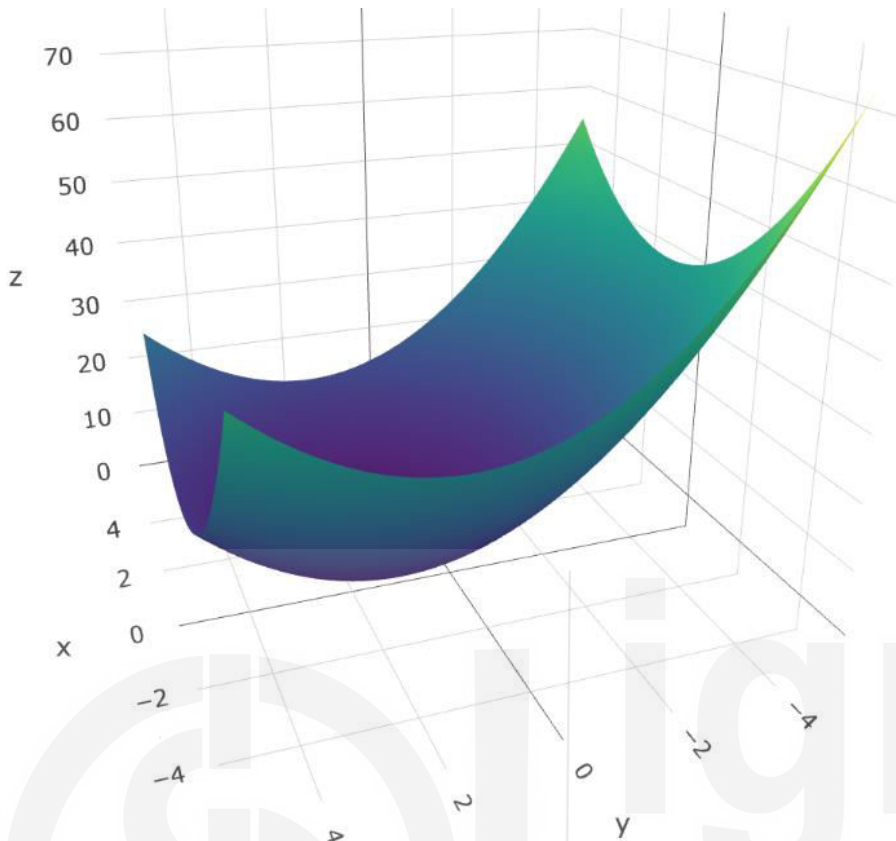


Fig. 13.10: Visualisation of the function given by (13.12) to get an idea of the location of the point(s) of local minima

From Fig. 13.10, we see that the point of local minima lies somewhere near about the point (1, 1). In fact, the function has global minima.

Iteration 1

Step 1: Here, we have to take an initial guess as the point $X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Step 2: $\nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 2 \end{bmatrix}$

$$\nabla f(X_0) = \nabla f(0, 0) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -S_0 \Rightarrow S_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Step 3: $H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so, $H_f(X_0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$$\begin{aligned}\text{Step 4: } \lambda_0 &= \frac{\mathbf{S}_0^T \mathbf{S}_0}{\mathbf{S}_0^T \mathbf{H}_f(\mathbf{X}_0) \mathbf{S}_0} = \frac{\begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}} = \frac{4 + 4}{\begin{bmatrix} 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}} \\ &= \frac{8}{8 + 8} = \frac{8}{16} = \frac{1}{2}\end{aligned}$$

$$\text{Step 5: } \mathbf{X}_1 = \mathbf{X}_0 + \lambda_0 \mathbf{S}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Step 6: Check whether the stopping criterion is met

$$\nabla f(\mathbf{X}_1) = \nabla f(1, 1) = \begin{bmatrix} 2(1) - 2 \\ 2(1) - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Great! The stopping criterion is met only in a single iteration. So, the point of local minima is (1, 1), which also meets our expectation mentioned earlier.

Example 7: Using the steepest descent method, obtain local minima of the function $f(x, y) = x^3 - 12x + y^3 - 27y + 40$ by taking an initial guess as the point (2, 1). The stopping criterion is $\|\nabla f(\mathbf{X}_k)\| < 0.4$.

Solution: The given function is $f(x, y) = x^3 - 12x + y^3 - 27y + 40$

This function has been visualised in Fig. 13.11.

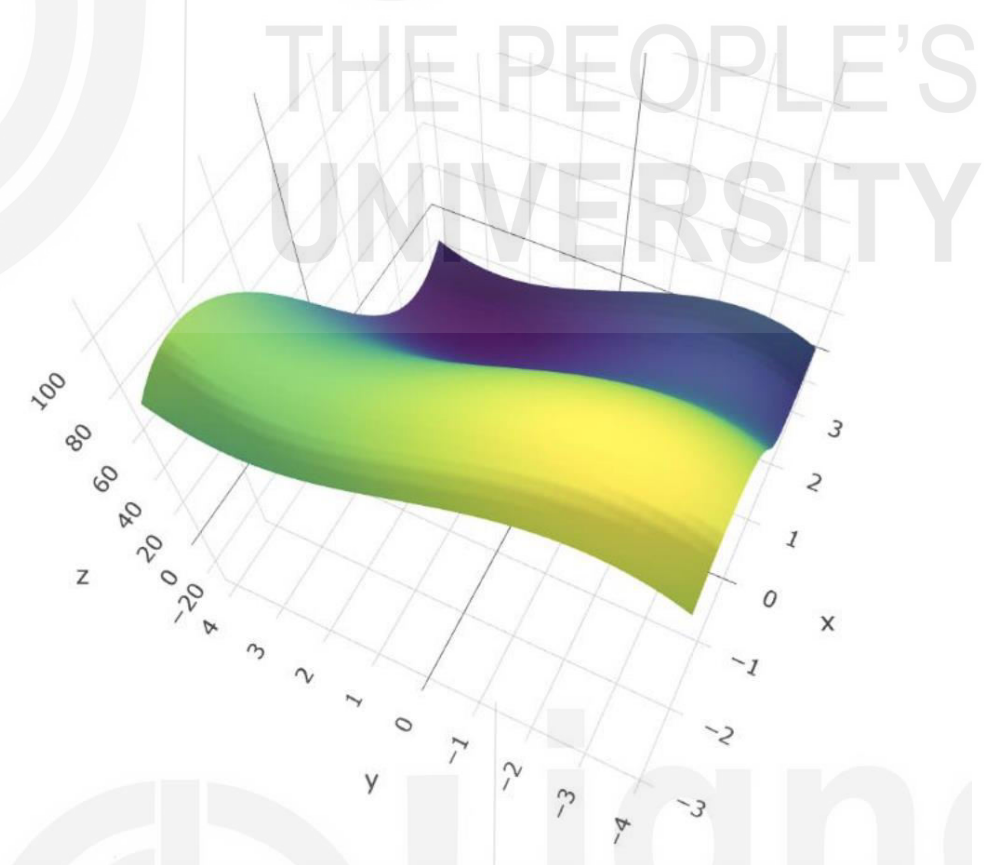


Fig. 13.11: Visualisation of the function

You may check that this function has four stationary points:

$$A(2, 3), B(2, -3), C(-2, 3), D(-2, -3).$$

Out of which point A is a point of local minima.

In this example, we will try to obtain the point of minima using the steepest descent method.

Iteration 1

Step 1: Here, we have to take an initial guess as the point $X_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$\text{Step 2: } \nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2 - 12 \\ 3y^2 - 27 \end{bmatrix}$$

$$\nabla f(X_0) = \nabla f(2, 1) = \begin{bmatrix} 0 \\ -24 \end{bmatrix} = -S_0 \Rightarrow S_0 = \begin{bmatrix} 0 \\ 24 \end{bmatrix}$$

$$\text{Step 3: } H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}, \text{ so, } H_f(X_0) = \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\begin{aligned} \text{Step 4: } \lambda_0 &= \frac{S_0^T S_0}{S_0^T H_f(X_0) S_0} = \frac{\begin{bmatrix} 0 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 24 \end{bmatrix}}{\begin{bmatrix} 0 & 24 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 24 \end{bmatrix}} = \frac{0 + 576}{\begin{bmatrix} 0 & 144 \end{bmatrix} \begin{bmatrix} 0 \\ 24 \end{bmatrix}} \\ &= \frac{576}{144 \times 24} = \frac{1}{6} \end{aligned}$$

$$\text{Step 5: } X_1 = X_0 + \lambda_0 S_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 0 \\ 24 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \dots (13.13)$$

Step 6: Check whether the stopping criterion is met

$$\nabla f(X_1) = \nabla f(2, 5) = \begin{bmatrix} 3(2)^2 - 12 \\ 3(5)^2 - 27 \end{bmatrix} = \begin{bmatrix} 0 \\ 48 \end{bmatrix}.$$

$$\|\nabla f(X_1)\| = \sqrt{0^2 + (48)^2} = 48. \text{ No as } 48 > 0.4.$$

Iteration 2

Step 1: After the first iteration, the initial guess is modified to the point:

$$X_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

$$\text{Step 2: } \nabla f(X_1) = \nabla f(2, 5) = \begin{bmatrix} 0 \\ 48 \end{bmatrix}. \text{ So, } S_1 = -\nabla f(X_1) = \begin{bmatrix} 0 \\ -48 \end{bmatrix}.$$

$$\text{Step 3: } H_f(X_1) = \begin{bmatrix} 12 & 0 \\ 0 & 30 \end{bmatrix}$$

$$\begin{aligned} \text{Step 4: } \lambda_1 &= \frac{S_1^T S_1}{S_1^T H_f(X_1) S_1} = \frac{\begin{bmatrix} 0 & -48 \end{bmatrix} \begin{bmatrix} 0 \\ -48 \end{bmatrix}}{\begin{bmatrix} 0 & -48 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 30 \end{bmatrix} \begin{bmatrix} 0 \\ -48 \end{bmatrix}} = \frac{48 \times 48}{\begin{bmatrix} 0 & -48 \times 30 \end{bmatrix} \begin{bmatrix} 0 \\ -48 \end{bmatrix}} \\ &= \frac{48 \times 48}{48 \times 30 \times 48} = \frac{1}{30} \end{aligned}$$

$$\text{Step 5: } X_2 = X_1 + \lambda_1 S_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \frac{1}{30} \begin{bmatrix} 0 \\ -48 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 5-\frac{8}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ 17/5 \end{bmatrix} \quad \dots (13.14)$$

Step 6: Check whether the stopping criterion is met

$$\begin{aligned} \nabla f(X_2) &= \nabla f(2, 17/5) = \begin{bmatrix} 3(2)^2 - 12 \\ 3(17/5)^2 - 27 \end{bmatrix} = \begin{bmatrix} 0 \\ 192/25 \end{bmatrix} = \begin{bmatrix} 0 \\ 7.68 \end{bmatrix}. \\ \|\nabla f(X_2)\| &= \sqrt{0^2 + (7.68)^2} = 7.68. \text{ No as } 7.68 > 0.4. \end{aligned}$$

Iteration 3

Step 1: After the second iteration, the initial guess is modified to the point

$$X_2 = \begin{bmatrix} 2 \\ 17/5 \end{bmatrix} \approx \begin{bmatrix} 2 \\ 3.4 \end{bmatrix}.$$

$$\text{Step 2: } \nabla f(X_2) = \nabla f(2, 17/5) = \begin{bmatrix} 0 \\ 192/25 \end{bmatrix}.$$

$$\text{So, } S_2 = -\nabla f(X_2) = \begin{bmatrix} 0 \\ -192/25 \end{bmatrix}.$$

$$\text{Step 3: } H_f(X_2) = \begin{bmatrix} 12 & 0 \\ 0 & 102/5 \end{bmatrix}$$

$$\begin{aligned} \text{Step 4: } \lambda_2 &= \frac{S_2^T S_2}{S_2^T H_f(X_2) S_2} = \frac{\begin{bmatrix} 0 & -192/25 \end{bmatrix} \begin{bmatrix} 0 \\ -192/25 \end{bmatrix}}{\begin{bmatrix} 0 & -192/25 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 30 \end{bmatrix} \begin{bmatrix} 0 \\ -192/25 \end{bmatrix}} \\ &= \frac{\frac{192}{25} \times \frac{192}{25}}{\begin{bmatrix} 0 & -\frac{192}{25} \times 30 \end{bmatrix} \begin{bmatrix} 0 \\ -192/25 \end{bmatrix}} = \frac{\frac{192}{25} \times \frac{192}{25}}{\frac{192}{25} \times 30 \times \frac{192}{25} \times 30} = \frac{1}{900} \end{aligned}$$

Step 5:

$$X_3 = X_2 + \lambda_2 S_2 = \begin{bmatrix} 2 \\ 17/5 \end{bmatrix} + \frac{1}{900} \begin{bmatrix} 0 \\ -192/25 \end{bmatrix} = \begin{bmatrix} 2+0 \\ \frac{17}{5} - \frac{32}{150 \times 5} \end{bmatrix} = \begin{bmatrix} 2 \\ 2518/750 \end{bmatrix}$$

... (13.15)

Step 6: Check whether the stopping criterion is met:

$$\nabla f(X_2) = \nabla f(2, 2518/750) = \begin{bmatrix} 3(2)^2 - 12 \\ 3(2518/750)^2 - 27 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3833472}{750 \times 750} \end{bmatrix} = \begin{bmatrix} 0 \\ 6.815 \end{bmatrix}.$$

$$\|\nabla f(X_4)\| = \sqrt{0^2 + (6.815)^2} = 6.815. \text{ No as } 6.815 > 0.4.$$

Iteration 4

Step 1: After the third iteration, the initial guess is modified to the point

$$X_3 = \begin{bmatrix} 2 \\ 2518/750 \end{bmatrix} \approx \begin{bmatrix} 2 \\ 3.3573 \end{bmatrix}.$$

Step 2: $\nabla f(X_3) = \nabla f(2, 2518/750) = \begin{bmatrix} 0 \\ \frac{3833472}{750 \times 750} \end{bmatrix}.$

So, $S_3 = -\nabla f(X_3) = \begin{bmatrix} 0 \\ -\frac{3833472}{750 \times 750} \end{bmatrix}.$

Step 3: $H_f(X_3) = \begin{bmatrix} 12 & 0 \\ 0 & 15108/750 \end{bmatrix}$

Step 4: $\lambda_3 = \frac{S_3^T S_3}{S_3^T H_f(X_3) S_3} = \frac{\begin{bmatrix} 0 & -\frac{3833472}{750 \times 750} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{3833472}{750 \times 750} \end{bmatrix}}{\begin{bmatrix} 0 & -\frac{3833472}{750 \times 750} \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 15108/750 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{3833472}{750 \times 750} \end{bmatrix}}$

$$= \frac{\frac{3833472}{750 \times 750} \times \frac{3833472}{750 \times 750}}{\frac{3833472}{750 \times 750} \times \frac{15108}{750} \times \frac{3833472}{750 \times 750}} = \frac{750}{15108} = \frac{1}{20.144}$$

Step 5: $X_4 = X_3 + \lambda_3 S_3 = \begin{bmatrix} 2 \\ 3.3573 \end{bmatrix} + \frac{1}{20.144} \begin{bmatrix} 0 \\ -\frac{3833472}{750 \times 750} \end{bmatrix}$

$$= \begin{bmatrix} 2+0 \\ 3.3573 - 0.3383 \end{bmatrix} = \begin{bmatrix} 2 \\ 3.019 \end{bmatrix}$$

... (13.16)

Step 6: Check whether the stopping criterion is met

$$\|\nabla f(X_4)\| = \sqrt{0^2 + (0.3431)^2} = 0.3431. \text{ Yes as } 0.3431 < 0.4.$$

Gradient Ascent

In this section, we have discussed the process of gradient descent. The gradient descent finds the minimum of a function using an iterative approach (Refer to equation 13.9). You start with an initial value X_0 and move along the steepest downward direction (because here we have $-\nabla f(X_0)$ in equation 13.9) till the point where the moving direction remains steepest descent. We continue the above iterative process till the time given stopping criterion is met. The Gradient ascent follows the same logic, except it finds the maximum of a function. Thus, in this case, $(-\nabla f(X_0))$ in equation 13.9 will just change in sign, as you need to find the steepest upward direction. Rest all the processes remain the same.

Stochastic Gradient Descent

Stochastic gradient descent is modelled on gradient descent. This is used when the function is of a higher dimension. Thus, in this case, it becomes computationally very expensive to compute the slope for all the nearby data points. Thus, to reduce the load on computation, a random number of near data points are selected, and an approximate optimisation is performed. This results in faster iterations.

Check Your Progress 2

- 1) In the gradient descent method, instead of selecting an optimal value of λ_k at each step, what are the possible problems that we may face if we take a fixed constant value of λ_k in each iteration?

.....
.....

- 2) Write one drawback of the gradient descent method

.....
.....

13.5 RANDOMISATION

Consider that you want to perform a study to ascertain which teaching method, say method A or method B, is better. You select students of class VII and divide them into two equivalent groups, say Group X and Group Y, and use teaching method A for Group X and teaching method B for Group Y. You compare the performance of students of the groups. Your study will give you a correct result if the students of Group X and Group Y have similar characteristics. This would be possible if you used randomisation.

Randomisation allows you to assign students to make two equivalent groups. Randomisation also removes any bias, thus enhancing the validity of your experiment.

One of the simplest ways of randomisation would be to generate a random number and use it to assign students to various groups. In fact, random numbers are very commonly used in data science to eliminate bias. Let us explain some of the basic features of random number generation in this section.

13.5.1 Random Numbers and Pseudo Random Numbers

The methodology of generating random numbers has a long history. The earliest methods were carried out by hand, throwing dice, dealing out well-shuffled cards, etc. Many lotteries are still operated in this way. In the early twentieth century, many mechanised devices were built to generate numbers more quickly. Later, electronic devices were used to generate random numbers. Rand Corporation (1955) generated a million numbers and they are available in table form. An easy method to read from the Rand Corporation table or some other statistical tables which contain a large number of random numbers is to select a row or a column of one, two or more digits. For example, Fisher and Yates (1963) Statistical Tables give 7500 two-figure random numbers arranged in six pages. Suppose one wants to select ten random numbers from 00-99, the simplest way of doing this is to select a row, column or diagonal of two-digit numbers randomly and read ten numbers from 00-99 as they appear in a column. If one wants ten uniformly distributed numbers $U(0, 1)$ then one can divide the chosen number by 99 (range of numbers between 00-99). For example, the first ten numbers of the first column of the Random Number Table contain two-digit numbers as:

03 97 16 12 55 16 84 63 33 57

These numbers are selected randomly (giving equal probabilities), without replacement, from two-digit numbers 00-99. If one wants to convert them to uniformly distributed $U(0, 1)$ variables, then one has to divide them by 99, which gives:

0.030 0.980 0.162 0.121 0.555 0.162 0.848 0.636 0.333 0.575

However, such methods are satisfactory when one requires a very small number of random numbers. But in most simulations, a very large number in thousands and millions are required. This requires a lot of memory, time and computers are not very efficient. For this purpose we require a procedure which should be fast and does not need much memory. Now modern computer use some inbuilt methods for generating. Pseudo Random Numbers (PRN) are not random numbers, but they behave like random numbers for all practical purposes. That is why they are called Pseudo Random Numbers. One may define PRN as: A sequence of PRN (U_i) is a deterministic sequence in the interval $[0, 1]$ having the same statistical properties as a sequence of random numbers. This means that any statistical test applied to a finite part of sequence (U_i), which aims to detect departure from randomness, would not reject the null hypothesis that the sequence consists of random numbers. Hence, we shall generate a deterministic sequence of PRN and then apply some relevant statistical tests to examine the hypothesis that the sequence is random. In case these tests do not reject this hypothesis, we shall take them as random numbers. But this may be noted here that they are not random numbers in the strict sense.

13.5.2 Random Number Generation

In the past, many methods have been used for generation of random numbers, but we shall not discuss all of them here and describe only the following two methods:

1. Middle Square Method
2. Linear Congruential Method

Middle Square Method

Computers are used for generating random numbers. With a computer, it is typically easier to generate random numbers by an arithmetic process. The method proposed for use on digital computers to generate random numbers is the “Middle Square Method”. In this method, if we want to generate n random numbers of r digits, then we take a random number of digits r , which is generated from any other method, then square this random number. If there are $2r$ digits in the square, then we take the middle r digits as the next random number. If there are less than $2r$ digits in the square, then we put zeroes in front of it to make $2r$ digits and then take the middle r digits. For example, if we want to generate a four-digit integer random number, then take a random number (which is generated by any other method), suppose 8937. To obtain the next number in this sequence, we square it. We get 79869969 and take the middle four digits 8699. So that the next random number generated is 8699. The next few random numbers in this sequence are 6726, 2390, 7121, ..., etc. If we have to generate a random number of two digits, then take a random number (which is generated by any other method), suppose the number is 13, then the square of this is 169. The square of this contains $(2r-1)$ digits, so put a single zero in front of this square to make $2r$ digits and then take middle two digits as the next random number, so that the next random number generated is 16. The few next random numbers generated in this sequence are 48, 30, 90, ..., etc.

The middle-square method has the following drawbacks:

1. This method tends to degenerate rapidly. A random number may reproduce itself. For example $x_9 = 7600$, $x_{10} = 7600$, $x_{11} = 7600$,...
2. If the number zero is ever generated, all subsequent numbers generated will also have a zero value unless steps are provided to handle this case.
3. A loop may get generated, i.e. the same sequence of random numbers can repeat. For example $x_{15} = 6100$, $x_{16} = 2100$, $x_{17} = 4100$, $x_{18} = 6100$, $x_{19} = 2100$.
4. This method is slow since many multiplications and divisions are required to access the middle digits in a fixed word binary computer.

Linear Congruential Generator (LCG) Method

A sequence of integers $z_1, z_2, \dots$ is generated by a recursive formula:

$$z_i = (az_{i-1} + c) \bmod m \quad \dots (13.17)$$

where m , a , and c are positive integers.

Equation (13.17) can also be written as:

$$z_i = (n) \bmod m \text{ by replacing } (az_{i-1} + c) \text{ by } n.$$

Here “ $(n) \bmod m$ ” means the remainder part of n/m .

Generally, “ $(n) \bmod m$ ” is always less than m .

For example, if $n = 4592$ and $m = 543$.

Then “ $(n) \bmod m$ ” = “ $(4592) \bmod 543$ ” = 248.

It is obvious that all z_i will satisfy:

$$0 \leq z_i \leq m-1$$

Uniformly distributed PRN from $U(0, 1)$ are obtained as u_i 's:

$$u_i = z_i/m$$

Whenever z_i takes the same value as taken earlier in the sequence, then the sequence is repeated again. The length of such a cycle is called the period (p). It is clear that $p \leq m$. For $p = m$, the LCG is said to be of full period. In actual simulations thousands of random numbers are required and it is desirable to have LCGs of large period, so that the same sequence is not repeated again.

Choice of Linear Congruential Generators

From the above discussion, it is clear that m should be as large as possible depending on the computer word size. When $c = 0$, the LCG is called a multiplicative generator. For a 32-bit computer, it has been shown that $m = 2^{31}-1$, $a = 7^5 = 16807$, $c = 0$ results in a good multiplicative generator:

$$z_i = (16807 z_{i-1} + 0) \bmod (2^{31}-1) \quad \dots (13.18)$$

Starting with initial arbitrary seed value z_0 , one can generate the required sequence of u_i 's.

For example, some uniform PRN from LCG given in (13.18) with $z_0 = 441$, $m = 2^{31}-1$

$$u_1 = z_1/m = 0.0034,$$

$$u_2 = z_2/m = 0.8063$$

Check Your Progress 3

- 1) Using the following LCG $x_i = (1573 x_{i-1} + 19) \bmod (10^3)$, obtain four x_i 's with $x_0 = 89$.

- 2) Using PRN generated in question 1, obtain four $U(0, 1)$ random variables.

13.6 SUMMARY

In this unit, we have covered the following points:

1. Basic terminology for optimisation
2. The concept of global and local optima
3. Use derivatives to find the extrema's
4. Extreme cases when extrema are cannot be addressed.
5. Use of Gradient descent to find the minima and use of gradient ascent to find maxima.
6. Define stochastic gradient descent.
7. Explain the concept of randomisation

13.7 SOLUTIONS / ANSWERS

Check Your Progress 1

- 1) (a) $f'(x) = 3x^2 - 6x = 3x(x-2)$. Therefore, critical points will be 0 and 2. After graphical analyses, you will get maximum at $x = 0$, with maximum value of the function as 2; and minimum at $x = 2$, with minimum value as -2 .
 - (b) $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x+1)(x-2)$. Critical points will be 0, -1 , and 2, with maximum at $x = 0$ and corresponding value of function as 2; local minima at $x = -1$ and global minima at $x = 2$ with corresponding value of function as -3 and -30 , respectively.
 - 2) a) Critical points are $x = 1$ and $x = 5$; Function attains relative maxima at $x = 1$ with relative maximum value of the function being $-14/3$; and it attains relative minima at $x = 5$ with relative minimum value of the function being $-16/3$.
 - b) Critical points are $x = 1$ and $x = -1$; Function attains relative maxima at $x = -1$ with relative maximum value of the function being -4 ; and it attains relative minima at $x = 1$ with relative minimum value of the function being 0.
- Note: You should be clear with the fact that relative minimum is not the minimum possible value of the function, and relative maximum is not the maximum possible value of the function.

Check Your Progress 2

- 1). If we take a fixed constant value of λ_k in each iteration, the possible problems may be:
 - If we take a fixed constant value of λ_k but a large value of λ_k then there may be the issue of overshooting. In other words, there may be a problem of missing a point of minima due to looping.
 - If we take a fixed constant value of λ_k but a small value of λ_k then it may take too long to reach the point of minima.

So, one of the best strategies is to optimise the value of λ_k at each iteration.

- 2). One of the possible drawbacks of the gradient descent method is that it may get stuck at a local minimum, which is not the global minimum.

Check Your Progress 3

- 1) We have $x_i = (1573x_{i-1} + 19) \bmod (10^3)$ with $x_0 = 89$

Therefore, $x_1 = 140016 \bmod (10^3) = 16$

$$x_2 = 25187 \bmod (10^3) = 187$$

$$x_3 = 294170 \bmod (10^3) = 170$$

$$x_4 = 267429 \bmod (10^3) = 429$$

- 2) We have $m = 10^3$ and $u_i = x_i/m$, then $u_1 = 0.016$, $u_2 = 0.187$, $u_3 = 0.170$, $u_4 = 0.429$

13.8 REFERENCES

IGNOU Course Material

1. Bachelor of Computer Applications (BCA_NEW), BCS-012: Mathematics Block 4: Vectors and Three-Dimensional Geometry, UNIT 4: Linear Programming (for Sections 13.2 and related portions in Sections 13.0 , 13.1, 13.6, 13.7 and Check Your Progress questions)
2. Bachelor of Arts (Honours) Economics (BAECH), BECC-102 Mathematical Methods for Economics – I, Block 4: Single Variable Optimisation, UNIT 12: Optimisation Methods (for Sections 13.3 and related portions in Sections 13.0 , 13.1, 13.6, 13.7 and Check Your Progress questions)
3. M.Sc. (Applied Statistics) (MSCAST), MST-011 Real Analysis, Calculus and Geometry, Block-2 Hyperplane and Calculus, Unit-7 Convex and Concave Function (for section 13.2 and related portions in Sections 13.0 , 13.1, 13.6, 13.7 and Check Your Progress questions)
4. M.Sc. (Applied Statistics) (MSCAST); MST-022 Linear Algebra and Multivariable Calculus; Block-4 Taylor Series, Optimisation and Gradient Descent; Unit-17 Constraint Optimisation and Gradient Descent (Section 13.4 and related portions in Sections 13.0 , 13.1, 13.6, 13.7 and Check Your Progress questions)
5. Post-Graduate Diploma in Applied Statistics (PGDAST), MST-005 Statistical Techniques, Block-4 Random Number Generation and Simulation Techniques, Unit-13 Random Number Generation for Discrete Variables (Section 13.5 and related portions in Sections 13.0 , 13.1, 13.6, 13.7 and Check Your Progress questions)

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TABLE I

COMMONLY USED VALUES OF STANDARD NORMAL VARIATE Z

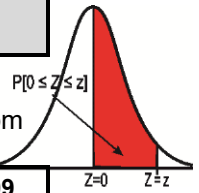
Confidence Interval	99%	98%	95%	90%
Level of Significance (α)	0.01 (1%)	0.02 (2%)	0.05 (5%)	0.10 (10%)
Two-Tailed	$\pm Z_{\alpha/2} = \pm 2.58$	$\pm Z_{\alpha/2} = \pm 2.33$	$\pm Z_{\alpha/2} = \pm 1.96$	$\pm Z_{\alpha/2} = \pm 1.645$
One (right)-Tailed	$Z_{\alpha} = 2.33$	$Z_{\alpha} = 2.05$	$Z_{\alpha} = 1.645$	$Z_{\alpha} = 1.28$
One (left)-Tailed	$-Z_{\alpha} = -2.33$	$-Z_{\alpha} = -2.05$	$-Z_{\alpha} = -1.645$	$-Z_{\alpha} = -1.28$



TABLE II

STANDARD NORMAL DISTRIBUTION (Z-TABLE)

The first column and first row of the table indicate the values of standard normal variate Z at first and second place of decimal. The entry represents the area $P[0 \leq Z \leq z]$ under the curve or probability from 0 to different values of Z .

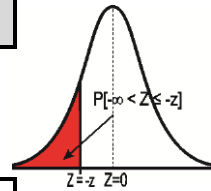


$\downarrow Z \rightarrow$	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3304	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

TABLE III

CUMULATIVE (NEGATIVE) STANDARD NORMAL TABLE

The first column and first row of the table indicate the values of standard normal variate Z at first and second place of decimal. The entry represents the area $P[-\infty < Z \leq -z]$ under the curve or probability from $-\infty$ to different negative values of Z .

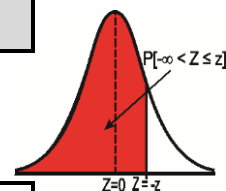


$\downarrow Z \rightarrow$	0.00	-0.01	-0.02	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
-3.6	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-4.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE IV

CUMULATIVE (POSITIVE) STANDARD NORMAL TABLE

The first column and first row of the table indicate the values of standard normal variate Z at first and second place of decimal. The entry represents the area $P[-\infty < Z \leq z]$ under the curve or probability from $-\infty$ to different positive values of Z .

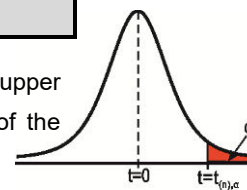


↓ Z →	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE V

STUDENT'S *t* DISTRIBUTION (t-TABLE)

The first column of this table indicates the degrees of freedom and the first row indicates a specified upper tail area (α). The entry represents the value of the *t*-statistic such that area under the curve of the *t*-distribution to its upper tail is equal to α .



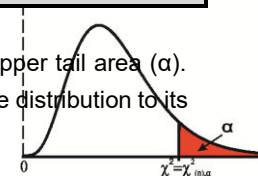
$\alpha =$	0.10	0.05	0.025	0.01	0.005
<i>n</i> (df) =1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
40	1.303	1.684	2.021	2.423	2.704
60	1.296	1.671	2.000	2.390	2.660
120	1.289	1.658	1.980	2.358	2.617
∞	1.282	1.645	1.960	2.326	2.576

Note: The area for negative values of *t*-statistic (test) is taken as the same as that for positive values, since the curve is symmetrical.

TABLE VI

CHI SQUARE DISTRIBUTION (CHI-SQUARE TABLE)

The first column of this table indicates the degrees of freedom and the first row indicates a specified upper tail area (α). The entry represents the value of the chi-square statistic such that area under the curve of the chi-square distribution to its upper tail is equal to α .



$\alpha =$	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
n(df) = 1	---	---	---	---	0.02	2.71	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	0.21	4.61	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	0.58	6.25	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.05	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	107.56	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81	140.17
120	83.85	86.92	91.58	95.70	100.62	140.23	146.57	152.21	158.95	163.64

TABLE VII

F DISTRIBUTION (F-TABLE)

F-table contains the values of the F-statistic for different set of degrees of freedom (n_1 , n_2) of numerator and denominator such that the area under the curve of the F-distribution to its right (upper tail) is equal to α . (F-values for $\alpha = 0.1$)

DF for Denominat or (n ₂)	Degrees of Freedom for Numerator (n ₁)																									
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	24	30	40	60	120	∞
1	39.86	49.50	53.60	55.83	57.23	58.21	58.91	59.44	59.86	60.20	60.47	60.70	60.91	61.07	61.22	61.35	61.47	61.57	61.66	61.74	62.00	62.26	62.53	62.79	63.06	63.33
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.40	9.41	9.41	9.42	9.42	9.43	9.43	9.44	9.44	9.44	9.45	9.46	9.47	9.47	9.48	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.22	5.21	5.20	5.20	5.20	5.19	5.19	5.19	5.18	5.18	5.17	5.16	5.15	5.14	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.96	3.94	3.92	3.91	3.90	3.89	3.88	3.87	3.86	3.86	3.85	3.85	3.84	3.83	3.82	3.80	3.79	3.78	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.28	3.27	3.26	3.25	3.24	3.23	3.22	3.22	3.21	3.21	3.19	3.17	3.16	3.14	3.12	3.11
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.92	2.90	2.89	2.88	2.87	2.86	2.86	2.85	2.84	2.84	2.82	2.80	2.78	2.76	2.74	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.79	2.75	2.72	2.70	2.68	2.67	2.65	2.64	2.63	2.62	2.61	2.61	2.60	2.59	2.58	2.56	2.54	2.51	2.49	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.52	2.50	2.49	2.48	2.46	2.45	2.45	2.44	2.43	2.42	2.40	2.38	2.36	2.34	2.32	2.29
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38	2.36	2.35	2.34	2.33	2.32	2.31	2.31	2.30	2.28	2.25	2.23	2.21	2.18	2.16
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.30	2.28	2.27	2.26	2.24	2.23	2.22	2.22	2.21	2.20	2.18	2.16	2.13	2.11	2.08	2.06
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.23	2.21	2.19	2.18	2.17	2.16	2.15	2.14	2.13	2.12	2.10	2.08	2.05	2.03	2.00	1.97
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.17	2.15	2.13	2.12	2.10	2.09	2.08	2.08	2.07	2.06	2.04	2.01	1.99	1.96	1.93	1.90
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.12	2.10	2.08	2.07	2.05	2.04	2.03	2.02	2.01	2.01	1.98	1.96	1.93	1.90	1.88	1.85
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.07	2.05	2.04	2.02	2.01	2.00	1.99	1.98	1.97	1.96	1.94	1.91	1.89	1.86	1.83	1.80
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.95	1.94	1.93	1.92	1.90	1.87	1.85	1.82	1.79	1.76
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	2.01	1.99	1.97	1.95	1.94	1.93	1.92	1.91	1.90	1.89	1.87	1.84	1.81	1.78	1.75	1.72
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.98	1.96	1.94	1.93	1.91	1.90	1.89	1.88	1.87	1.86	1.84	1.81	1.78	1.75	1.72	1.69
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.95	1.93	1.92	1.90	1.89	1.87	1.86	1.85	1.85	1.84	1.81	1.78	1.75	1.72	1.69	1.66
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.93	1.91	1.89	1.88	1.86	1.85	1.84	1.83	1.82	1.81	1.79	1.76	1.73	1.70	1.67	1.63
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.91	1.89	1.87	1.86	1.84	1.83	1.82	1.81	1.80	1.79	1.77	1.74	1.71	1.68	1.64	1.61
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.90	1.88	1.86	1.84	1.83	1.81	1.80	1.79	1.78	1.78	1.75	1.72	1.69	1.66	1.62	1.59
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.88	1.86	1.84	1.83	1.81	1.80	1.79	1.78	1.77	1.76	1.73	1.70	1.67	1.64	1.60	1.57
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.87	1.85	1.83	1.81	1.80	1.78	1.77	1.76	1.75	1.74	1.72	1.69	1.66	1.62	1.59	1.55
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.85	1.83	1.81	1.80	1.78	1.77	1.76	1.75	1.74	1.73	1.70	1.67	1.64	1.61	1.57	1.53
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.84	1.82	1.80	1.79	1.77	1.76	1.75	1.74	1.73	1.72	1.69	1.66	1.63	1.59	1.56	1.52
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.83	1.81	1.79	1.77	1.76	1.75	1.73	1.72	1.71	1.71	1.68	1.65	1.61	1.58	1.54	1.50
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.82	1.80	1.78	1.76	1.75	1.74	1.72	1.71	1.70	1.70	1.67	1.64	1.60	1.57	1.53	1.49
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.81	1.79	1.77	1.75	1.74	1.73	1.71	1.70	1.69	1.69	1.66	1.63	1.59	1.56	1.52	1.48
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.80	1.78	1.76	1.75	1.73	1.72	1.71	1.69	1.68	1.68	1.65	1.62	1.58	1.55	1.51	1.47
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.79	1.77	1.75	1.74	1.72	1.71	1.70	1.69	1.68	1.67	1.64	1.61	1.57	1.54	1.50	1.46
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.74	1.71	1.70	1.68	1.66	1.65	1.64	1.62	1.61	1.61	1.57	1.54	1.51	1.47	1.42	1.38
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.68	1.66	1.64	1.62	1.60	1.59	1.58	1.56	1.55	1.54	1.51	1.48	1.44	1.40	1.35	1.29
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.63	1.60	1.58	1.56	1.55	1.53	1.52	1.50	1.49	1.48	1.45	1.41	1.37	1.32	1.26	1.19
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.57	1.55	1.52	1.51	1.49	1.47	1.46	1.44	1.43	1.42	1.38	1.34	1.30	1.24	1.17	1.00

F values for $\alpha = 0.05$

DF for Denominator (n ₂)	Degrees of Freedom for Numerator (n ₁)																									
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	24	30	40	60	120	∞
1	161	199	216	225	230	234	237	239	240	242	243	244	245	245	246	246	247	247	248	248	249	250	251	252	253	254
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.39	19.40	19.40	19.41	19.42	19.43	19.43	19.43	19.44	19.44	19.44	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.72	8.70	8.69	8.68	8.67	8.67	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83	5.82	5.81	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59	4.58	4.57	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91	3.90	3.88	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48	3.47	3.46	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19	3.17	3.16	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97	2.96	2.95	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81	2.80	2.79	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69	2.67	2.66	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58	2.57	2.56	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50	2.48	2.47	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43	2.41	2.40	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37	2.35	2.34	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32	2.30	2.29	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27	2.26	2.24	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23	2.22	2.20	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.18	2.17	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17	2.15	2.14	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12	2.11	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11	2.10	2.08	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.24	2.20	2.18	2.15	2.13	2.11	2.09	2.08	2.06	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18	2.15	2.13	2.11	2.09	2.07	2.05	2.04	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05	2.04	2.02	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03	2.02	2.00	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.17	2.13	2.10	2.08	2.06	2.04	2.02	2.00	1.99	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00	1.99	1.97	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99	1.97	1.96	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98	1.96	1.95	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00	1.97	1.95	1.92	1.90	1.89	1.87	1.85	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80	1.78	1.76	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.87	1.83	1.80	1.78	1.75	1.73	1.71	1.69	1.67	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75	1.72	1.69	1.67	1.64	1.62	1.60	1.59	1.57	1.52	1.46	1.39	1.32	1.22	1.00

F values for $\alpha = 0.025$

DF for Denominat or (n ₂)	Degrees of Freedom for Numerator (n ₁)																										
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	24	30	40	60	120	∞	
1	648	800	864	900	922	937	948	957	963	969	973	977	980	983	985	987	989	990	992	993	997	1001	1006	1010	1014	1018	
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.41	39.42	39.43	39.43	39.44	39.44	39.44	39.45	39.45	39.46	39.47	39.47	39.48	39.49	39.50	
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.37	14.34	14.30	14.28	14.25	14.23	14.21	14.20	14.18	14.17	14.12	14.08	14.04	13.99	13.95	13.90	
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.79	8.75	8.72	8.68	8.66	8.63	8.61	8.59	8.58	8.56	8.51	8.46	8.41	8.36	8.31	8.26	
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.57	6.52	6.49	6.46	6.43	6.40	6.38	6.36	6.34	6.33	6.28	6.23	6.18	6.12	6.07	6.02	
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.41	5.37	5.33	5.30	5.27	5.24	5.22	5.20	5.18	5.17	5.12	5.07	5.01	4.96	4.90	4.85	
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.71	4.67	4.63	4.60	4.57	4.54	4.52	4.50	4.48	4.47	4.42	4.36	4.31	4.25	4.20	4.14	
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.24	4.20	4.16	4.13	4.10	4.08	4.05	4.03	4.02	4.00	3.95	3.89	3.84	3.78	3.73	3.67	
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.91	3.87	3.83	3.80	3.77	3.74	3.72	3.70	3.68	3.67	3.61	3.56	3.51	3.45	3.39	3.33	
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.66	3.62	3.58	3.55	3.52	3.50	3.47	3.45	3.44	3.42	3.37	3.31	3.26	3.20	3.14	3.08	
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.47	3.43	3.39	3.36	3.33	3.30	3.28	3.26	3.24	3.23	3.17	3.12	3.06	3.00	2.94	2.88	
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.32	3.28	3.24	3.21	3.18	3.15	3.13	3.11	3.09	3.07	3.02	2.96	2.91	2.85	2.79	2.73	
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.20	3.15	3.12	3.08	3.05	3.03	3.00	2.98	2.96	2.95	2.89	2.84	2.78	2.72	2.66	2.60	
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.09	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86	2.84	2.79	2.73	2.67	2.61	2.55	2.49	
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	3.01	2.96	2.92	2.89	2.86	2.84	2.81	2.79	2.77	2.76	2.70	2.64	2.59	2.52	2.46	2.40	
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.68	2.63	2.57	2.51	2.45	2.38	2.32	
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.87	2.82	2.79	2.75	2.72	2.70	2.67	2.65	2.63	2.62	2.56	2.50	2.44	2.38	2.32	2.25	
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.81	2.77	2.73	2.70	2.67	2.64	2.62	2.60	2.58	2.56	2.50	2.45	2.38	2.32	2.26	2.19	
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.76	2.72	2.68	2.65	2.62	2.59	2.57	2.55	2.53	2.51	2.45	2.39	2.33	2.27	2.20	2.13	
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.72	2.68	2.64	2.60	2.57	2.55	2.52	2.50	2.48	2.46	2.41	2.35	2.29	2.22	2.16	2.09	
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.68	2.64	2.60	2.56	2.53	2.51	2.48	2.46	2.44	2.42	2.37	2.31	2.25	2.18	2.11	2.04	
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.65	2.60	2.56	2.53	2.50	2.47	2.45	2.43	2.41	2.39	2.33	2.27	2.21	2.15	2.08	2.00	
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.62	2.57	2.53	2.50	2.47	2.44	2.42	2.39	2.37	2.36	2.30	2.24	2.18	2.11	2.04	1.97	
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.59	2.54	2.50	2.47	2.44	2.41	2.39	2.36	2.35	2.33	2.27	2.21	2.15	2.08	2.01	1.94	
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.56	2.51	2.48	2.44	2.41	2.38	2.36	2.34	2.32	2.30	2.24	2.18	2.12	2.05	1.98	1.91	
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.54	2.49	2.45	2.42	2.39	2.36	2.34	2.31	2.29	2.28	2.22	2.16	2.09	2.03	1.95	1.88	
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.51	2.47	2.43	2.39	2.36	2.34	2.31	2.29	2.27	2.25	2.19	2.13	2.07	2.00	1.93	1.85	
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.49	2.45	2.41	2.37	2.34	2.32	2.29	2.27	2.25	2.23	2.17	2.11	2.05	1.98	1.91	1.83	
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.46	2.48	2.43	2.39	2.36	2.30	2.27	2.25	2.23	2.21	2.15	2.09	2.03	1.96	1.89	1.81	
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.33	2.41	2.37	2.34	2.31	2.28	2.26	2.23	2.21	2.20	2.14	2.07	2.01	1.94	1.87	1.79	
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.26	2.29	2.25	2.21	2.18	2.15	2.13	2.11	2.09	2.07	2.01	1.94	1.88	1.80	1.72	1.64	
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.22	2.17	2.13	2.09	2.06	2.03	2.01	1.98	1.96	1.94	1.88	1.82	1.74	1.67	1.58	1.48	
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.10	2.05	2.01	1.98	1.94	1.92	1.89	1.87	1.84	1.82	1.76	1.69	1.61	1.53	1.43	1.31	
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.99	1.94	1.90	1.87	1.83	1.80	1.78	1.75	1.73	1.71	1.64	1.57	1.48	1.39	1.27	1.00	

F values for $\alpha = 0.01$

DF for Denominat or (n_2)	Degrees of Freedom for Numerator (n_1)																									
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	24	30	40	60	120	∞
1	4063	4992	5404	5637	5760	5890	5890	6025	6025	6025	6025	6167	6167	6167	6167	6167	6167	6167	6167	6167	6235	6261	6287	6313	6339	6366
2	98.50	99.00	99.15	99.27	99.30	99.34	99.34	99.38	99.38	99.38	99.42	99.42	99.42	99.42	99.42	99.42	99.46	99.46	99.46	99.46	99.46	99.47	99.47	99.48	99.49	99.50
3	34.11	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.34	27.23	27.13	27.05	26.98	26.92	26.87	26.83	26.79	26.75	26.72	26.69	26.60	26.51	26.41	26.32	26.22	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.45	14.37	14.31	14.25	14.20	14.15	14.11	14.08	14.05	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.96	9.89	9.83	9.77	9.72	9.68	9.64	9.61	9.58	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72	7.66	7.60	7.56	7.52	7.48	7.45	7.42	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47	6.41	6.36	6.31	6.28	6.24	6.21	6.18	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67	5.61	5.56	5.52	5.48	5.44	5.41	5.38	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11	5.05	5.01	4.96	4.92	4.89	4.86	4.83	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71	4.65	4.60	4.56	4.52	4.49	4.46	4.43	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40	4.34	4.29	4.25	4.21	4.18	4.15	4.12	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16	4.10	4.05	4.01	3.97	3.94	3.91	3.88	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96	3.91	3.86	3.82	3.78	3.75	3.72	3.69	3.66	3.59	3.51	3.43	3.34	3.26	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.86	3.80	3.75	3.70	3.66	3.62	3.59	3.56	3.53	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67	3.61	3.56	3.52	3.49	3.45	3.42	3.40	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55	3.50	3.45	3.41	3.37	3.34	3.31	3.28	3.26	3.18	3.10	3.02	2.93	2.85	2.75
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46	3.40	3.35	3.31	3.27	3.24	3.21	3.19	3.16	3.08	3.00	2.92	2.84	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37	3.32	3.27	3.23	3.19	3.16	3.13	3.10	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30	3.24	3.19	3.15	3.12	3.08	3.05	3.03	3.00	2.93	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23	3.18	3.13	3.09	3.05	3.02	2.99	2.96	2.94	2.86	2.78	2.70	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.24	3.17	3.12	3.07	3.03	2.99	2.96	2.93	2.90	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12	3.07	3.02	2.98	2.94	2.91	2.88	2.85	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.14	3.07	3.02	2.97	2.93	2.89	2.86	2.83	2.80	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03	2.98	2.93	2.89	2.85	2.82	2.79	2.76	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	3.06	2.99	2.94	2.89	2.85	2.81	2.78	2.75	2.72	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96	2.90	2.86	2.82	2.78	2.75	2.72	2.69	2.66	2.59	2.50	2.42	2.33	2.23	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.99	2.93	2.87	2.82	2.78	2.75	2.71	2.68	2.66	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90	2.84	2.79	2.75	2.72	2.68	2.65	2.63	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.93	2.87	2.81	2.77	2.73	2.69	2.66	2.63	2.60	2.57	2.50	2.41	2.33	2.23	2.14	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84	2.79	2.74	2.70	2.66	2.63	2.60	2.57	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66	2.61	2.56	2.52	2.48	2.45	2.42	2.39	2.37	2.29	2.20	2.11	2.02	1.92	1.81
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50	2.44	2.39	2.35	2.31	2.28	2.25	2.22	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.40	2.34	2.28	2.23	2.19	2.15	2.12	2.09	2.06	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.19	2.13	2.08	2.04	2.00	1.97	1.94	1.91	1.88	1.79	1.70	1.59	1.47	1.33	1.00

F values for $\alpha = 0.005$

DF for Denominator (n ₂)	Degrees of Freedom for Numerator (n ₁)																									
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	24	30	40	60	120	∞
1	16211	19999	21615	22500	23056	23437	23715	23925	24091	24224	24334	24426	24505	24572	24630	24681	24727	24767	24803	24836	24940	25044	25148	25253	25359	25464
2	198.50	199.00	199.17	199.25	199.30	199.33	199.36	199.37	199.39	199.40	199.41	199.42	199.42	199.43	199.43	199.44	199.44	199.44	199.45	199.45	199.46	199.47	199.47	199.48	199.49	199.50
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.52	43.39	43.27	43.17	43.08	43.01	42.94	42.88	42.83	42.78	42.62	42.47	42.31	42.15	41.99	41.83
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.82	20.70	20.60	20.51	20.44	20.37	20.31	20.26	20.21	20.17	20.03	19.89	19.75	19.61	19.47	19.32
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.49	13.38	13.29	13.21	13.15	13.09	13.03	12.98	12.94	12.90	12.78	12.66	12.53	12.40	12.27	12.14
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.13	10.03	9.95	9.88	9.81	9.76	9.71	9.66	9.62	9.59	9.47	9.36	9.24	9.12	9.00	8.88
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.27	8.18	8.10	8.03	7.97	7.91	7.87	7.83	7.79	7.75	7.64	7.53	7.42	7.31	7.19	7.08
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.10	7.01	6.94	6.87	6.81	6.76	6.72	6.68	6.64	6.61	6.50	6.40	6.29	6.18	6.06	5.95
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.31	6.23	6.15	6.09	6.03	5.98	5.94	5.90	5.86	5.83	5.73	5.62	5.52	5.41	5.30	5.19
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.75	5.66	5.59	5.53	5.47	5.42	5.38	5.34	5.31	5.27	5.17	5.07	4.97	4.86	4.75	4.64
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.32	5.24	5.16	5.10	5.05	5.00	4.96	4.92	4.89	4.86	4.76	4.65	4.55	4.45	4.34	4.23
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.99	4.91	4.84	4.77	4.72	4.67	4.63	4.59	4.56	4.53	4.43	4.33	4.23	4.12	4.01	3.90
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.72	4.64	4.57	4.51	4.46	4.41	4.37	4.33	4.30	4.27	4.17	4.07	3.97	3.87	3.76	3.65
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.51	4.43	4.36	4.30	4.25	4.20	4.16	4.12	4.09	4.06	3.96	3.86	3.76	3.66	3.55	3.44
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.33	4.25	4.18	4.12	4.07	4.02	3.98	3.95	3.91	3.88	3.79	3.69	3.58	3.48	3.37	3.26
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.18	4.10	4.03	3.97	3.92	3.87	3.83	3.80	3.76	3.73	3.64	3.54	3.44	3.33	3.22	3.11
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	4.05	3.97	3.90	3.84	3.79	3.75	3.71	3.67	3.64	3.61	3.51	3.41	3.31	3.21	3.10	2.98
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.94	3.86	3.79	3.73	3.68	3.64	3.60	3.56	3.53	3.50	3.40	3.30	3.20	3.10	2.99	2.87
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.84	3.76	3.70	3.64	3.59	3.54	3.50	3.46	3.43	3.40	3.31	3.21	3.11	3.00	2.89	2.78
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.76	3.68	3.61	3.55	3.50	3.46	3.42	3.38	3.35	3.32	3.22	3.12	3.02	2.92	2.81	2.69
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.68	3.60	3.54	3.48	3.43	3.38	3.34	3.31	3.27	3.24	3.15	3.05	2.95	2.84	2.73	2.61
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.61	3.54	3.47	3.41	3.36	3.31	3.27	3.24	3.21	3.18	3.08	2.98	2.88	2.77	2.66	2.55
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.55	3.47	3.41	3.35	3.30	3.25	3.21	3.18	3.15	3.12	3.02	2.92	2.82	2.71	2.60	2.48
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.50	3.42	3.35	3.30	3.25	3.20	3.16	3.12	3.09	3.06	2.97	2.87	2.77	2.66	2.55	2.43
25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.45	3.37	3.30	3.25	3.20	3.15	3.11	3.08	3.04	3.01	2.92	2.82	2.72	2.61	2.50	2.38
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.40	3.33	3.26	3.20	3.15	3.11	3.07	3.03	3.00	2.97	2.87	2.77	2.67	2.56	2.45	2.33
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.36	3.28	3.22	3.16	3.11	3.07	3.03	2.99	2.96	2.93	2.83	2.73	2.63	2.52	2.41	2.29
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.32	3.25	3.18	3.12	3.07	3.03	2.99	2.95	2.92	2.89	2.79	2.69	2.59	2.48	2.37	2.25
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.29	3.21	3.15	3.09	3.04	2.99	2.95	2.92	2.88	2.86	2.76	2.66	2.56	2.45	2.33	2.21
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.25	3.18	3.11	3.06	3.01	2.96	2.92	2.89	2.85	2.82	2.73	2.63	2.52	2.42	2.30	2.18
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	3.03	2.95	2.89	2.83	2.78	2.74	2.70	2.66	2.63	2.60	2.50	2.40	2.30	2.18	2.06	1.93
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.82	2.74	2.68	2.62	2.57	2.53	2.49	2.45	2.42	2.39	2.29	2.19	2.08	1.96	1.83	1.69
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.62	2.54	2.48	2.42	2.37	2.33	2.29	2.25	2.22	2.19	2.09	1.98	1.87	1.75	1.61	1.43
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.43	2.36	2.29	2.24	2.19	2.14	2.10	2.06	2.03	2.00	1.90	1.79	1.67	1.53	1.36	1.36